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Properties of Morse Forms that Determine Compact Foliations on M_g^2

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In [1, 2] P. Arnoux and G. Levitt showed that the topology of the foliation of a Morse form ω on a compact manifold is closely related to the structure of the integration mapping $[\omega]\colon H_1(M) \to \mathbb{R}$. In this paper we consider the foliation of a Morse form on a two-dimensional manifold M_g^2. We study the relationship of the subgroup $\operatorname{Ker}[\omega] \subset H_1(M_g^2)$ with the topology of the foliation. We consider the structure of the subgroup $\operatorname{Ker}[\omega]$ for a compact foliation and prove a criterion for the compactness of a foliation.

§1. Preliminary definitions

Consider a closed form ω with Morse singularities on M_g^2. This form determines a foliation $\mathcal{F}$ on $M_g^2 \setminus \operatorname{Sing}\omega$.

Let us define a foliation with singularities $\mathcal{F}_\omega$ on M_g^2 as follows.

Suppose that the foliation $\mathcal{F}$ is locally (in a sufficiently small neighborhood of a singular point $p \in \operatorname{Sing}\omega$) determined by the levels of a function f_p such that $f_p(p) = 0$.

Definition 1. A *nonsingular leaf of a foliation* $\mathcal{F}_\omega$ is a leaf $\gamma \in \mathcal{F}$ such that $\gamma \cap f_p^{-1}(0) = \varnothing$ for all $p \in \operatorname{Sing}\omega$.

Put $F_p = p \cup \{\gamma \in \mathcal{F} \mid \gamma \cap f_p^{-1}(0) \neq \varnothing\}$. Also put $F = \bigcup_{p\in\operatorname{Sing}\omega} F_p$.

Definition 2. A *singular leaf of a foliation* $\mathcal{F}_\omega$ is a connected component of F.

There is only a finite number of singular leaves (because the form is Morse).

A foliation $\mathcal{F}_\omega$ is called *compact* if all its leaves are compact.

A closed form ω determines the mapping $[\omega]\colon H_1(M_g^2) \to \mathbb{R}$ (integration over cycles). The image of this mapping $\operatorname{Im}[\omega]$ represents the period group of the form ω. Note that $\operatorname{rk}\operatorname{Im}[\omega] = \operatorname{dirr}\omega + 1$, where $\operatorname{dirr}\omega$ is the degree of irrationality of the form ω.

If $\operatorname{dirr}\omega \le 0$, then the foliation $\mathcal{F}_\omega$ is compact [3]. If $\operatorname{dirr}\omega \ge g$, then the foliation $\mathcal{F}_\omega$ has a noncompact leaf [4]. If $0 < \operatorname{dirr}\omega < g$, then the foliation can be compact as well as noncompact. The study of the subgroup $\operatorname{Ker}[\omega]$ yields a condition for the compactness of a foliation in the latter case also.

Consider the intersection operation of 1-cycles

$$\varphi\colon H_1(M_g^2) \times H_1(M_g^2) \to \mathbb{Z}.$$

This operation is a nondegenerate skew-symmetric bilinear mapping.

By φ_ω denote the restriction of the mapping φ to the subgroup $\operatorname{Ker}[\omega] \subset H_1(M_g^2)$:

$$\varphi_\omega\colon \operatorname{Ker}[\omega] \times \operatorname{Ker}[\omega] \to \mathbb{Z}.$$

Obviously, $\operatorname{rk}\operatorname{Ker}\varphi_\omega \le \operatorname{rk}\operatorname{Ker}[\omega] = 2g - (\operatorname{dirr}\omega + 1)$. For small values of $\operatorname{dirr}\omega$ a sharper estimate exists.

Proposition 1. $\operatorname{rk}\operatorname{Ker}\varphi_\omega \le \operatorname{dirr}\omega + 1$.

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